Section 5.5 The Fundamental Theorem of Calculus part 2
(Minimum Homework: all odds)

In section 5.4 we learned how to use the Fundamental Theorem of Calculus to calculate the "NET" area of the between graph of a function and the x-axis.

We did this problem as an example:

Example: Use the Fundamental Theorem of Calculus to evaluate the definite integral.

$\int_{-2}^{2} x\,dx$

Step 1: Perform the integration. Use this rule: Power Rule:

$\int_{-2}^{2} x\,dx = \frac{1}{2}x^2 \Big|_{-2}^{2}$

Step 2: Evaluate the integral at 2, then at -2

$\frac{1}{2}(2)^2 = \frac{1}{2} * 4 = 2$

$\frac{1}{2}(-2)^2 = \frac{1}{2} * 4 = 2$

Step 3: Subtract the results to determine the area

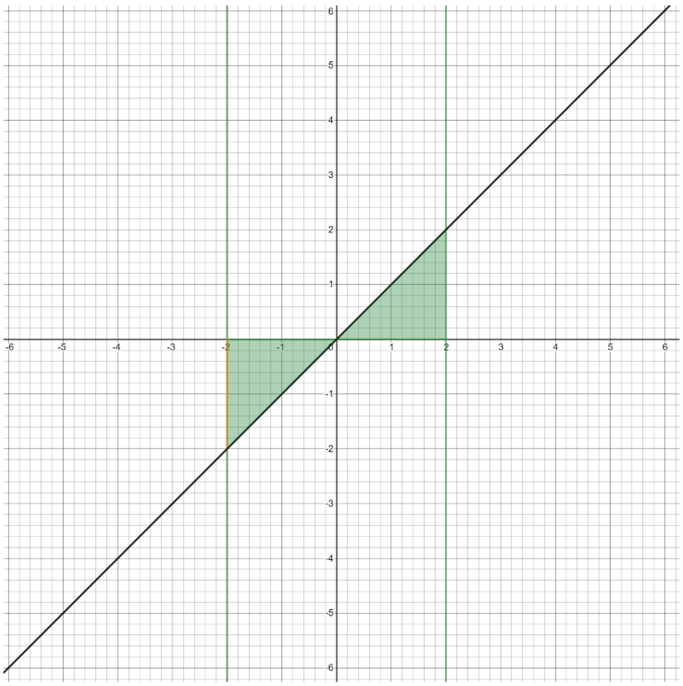
$\int_{-2}^{2} x\,dx = 2 - 2 = 0$

Answer: $\int_{-2}^{2} x\,dx = 0$

The integral equaled zero as there were equal parts of the graph above and below the x-axis.

Here is a graph of the function that we just found the definite integral for

$f(x) = x \quad with\ shading\ between - 2, and\ 2$



The graph produces 2 triangles that have the same area.
Both triangles have a base of 2 and a height of 2

Top triangle area: $\frac{2*2}{2} = 2\ square\ units$

Bottom triangle area: $\frac{2*2}{2} = 2\ square\ units$

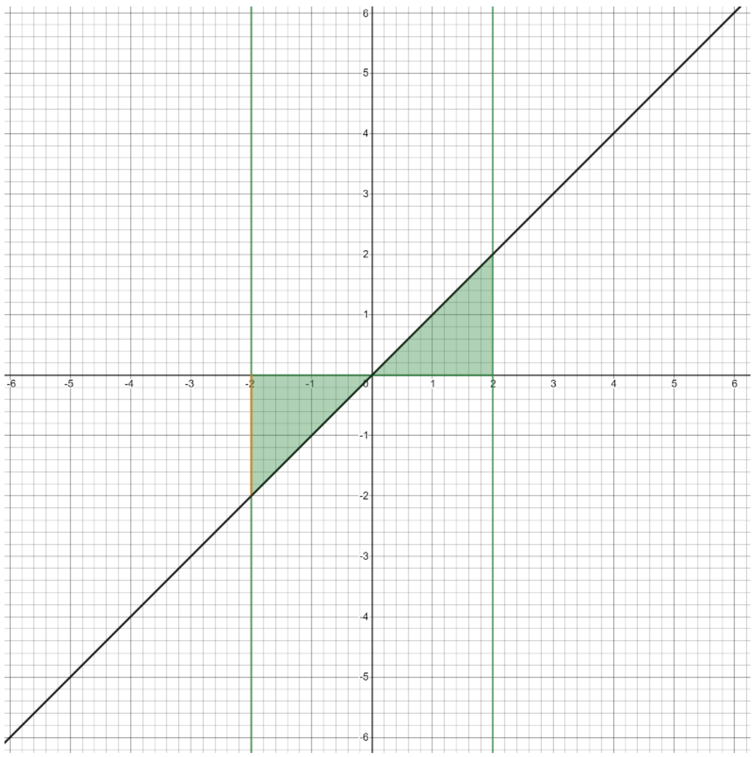
The integral use to compute the shaded area will equal 0.

$\int_{-2}^{2} x\,dx = 0$

The bottom triangle's area will be considered -2, and the top triangle's area $+2$. The net area will be 0.

In section 5.4 we calculated NET area.

In section 5.5 we want to make each area positive and find a total shaded area.



Section 5.4: NET area = $\int_{-2}^{2} x\, dx = 0$

Section 5.5: Combined area: $2 + 2 = 4$

Top triangle area: $\frac{2*2}{2} = 2\ square\ units$

Bottom triangle area: $\frac{2*2}{2} = 2\ square\ units$

We need to be able to find the combined area using Calculus.

This is not so hard.

We just calculate each area separately (making any negative area positive) and then add the areas together.

For our example this is how we find the combined area between the $x - axis$ and $f(x) = x$ over the interval $[-2,2]$ using definite integrals.

First

- Find the area of the portion of the graph below the x-axis.
- This area will be negative.
- We will need to make it positive.

The left triangle starts at $x = -2$ and ends at $x = 0$

This calculation will find the area and then make it positive.

$$\left|\int_{-2}^{0} x dx\right|$$

Find the integral

$$= \left|\frac{1}{2}x^2 \Big|_{-2}^{0}\right|$$

Evaluate the integral first at 0 then at -2

$$\frac{1}{2}(0)^2 = 0$$

$$\frac{1}{2}(-2)^2 = \frac{1}{2} * 4 = 2$$

Subtract to find the value of the integral, then apply the absolute value to make the area positive.

$$\left|\int_{-2}^{0} x dx\right| = |0 - 2| = |-2| = 2$$

(The value of the integral is -2. We need to make the area positive. So we do not net out back to zero.)

Second

- Find the area of the portion of the graph above the x-axis.
- This area will be positive.
- We will leave it positive, and no absolute values will be needed.

The right triangle starts at $x = 0$ and ends at $x = 2$

This calculation will find the area.(no absolute values are needed as this calculation will be positive)

$\int_0^2 xdx$

Find the integral

$= \frac{1}{2}x^2|_0^2$

Evaluate the integral first at 2 then at 0

$\frac{1}{2}(2)^2 = \frac{1}{2} * 4 = 2$

$\frac{1}{2}(0)^2 = 0$

Subtract to find the value of the integral

$\int_0^2 xdx = 2 - 0 = 2$

The right triangle also has an area of 2 square units.

Calculate the total shaded area

Area of the lower triangle + Area of the upper triangle

$= 2 + 2 = 4$

Answer: Shaded area is 4 square units.

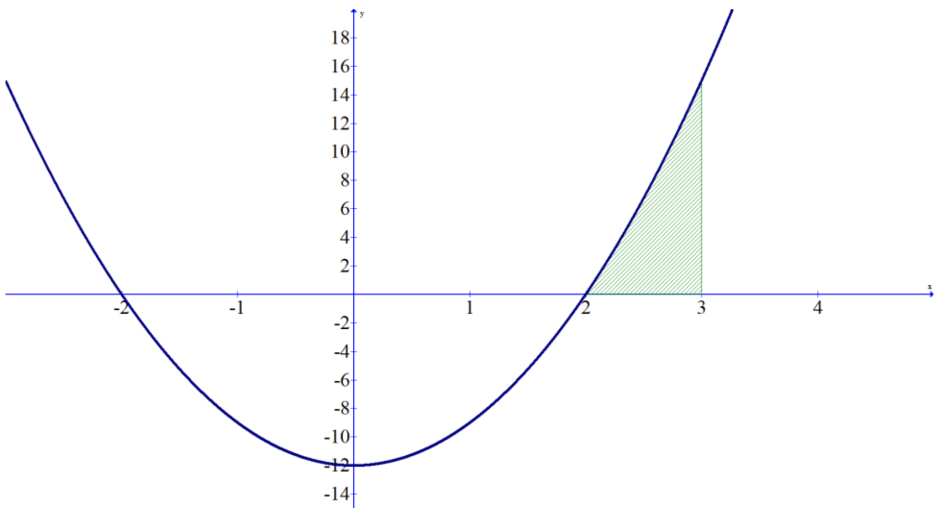
Just to emphasize:

NET area = $\int_{-2}^{2} x\, dx = 0$

Combined area = $\left|\int_{-2}^{0} xdx\right| + \int_0^2 xdx = |-2| + 2 = 2 + 2 = 4$

Example: The function $f(x) = 3x^2 - 12$ *is graphed below*.

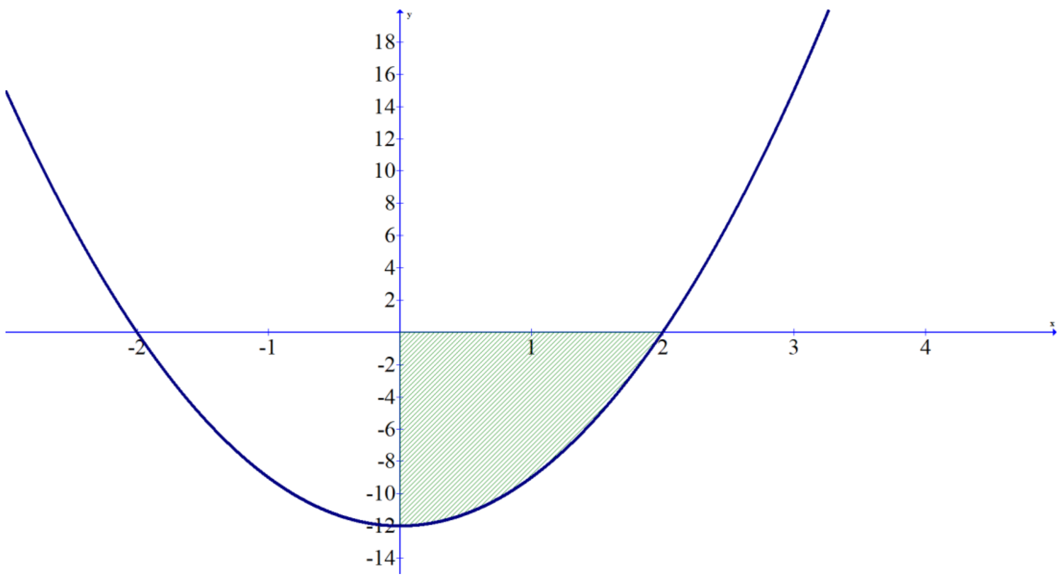
a) Use integration on your calculator to determine the area shaded between x = 2 and x = 3



Answer part a: $\int_2^3 (3x^2 - 12)dx = 7$

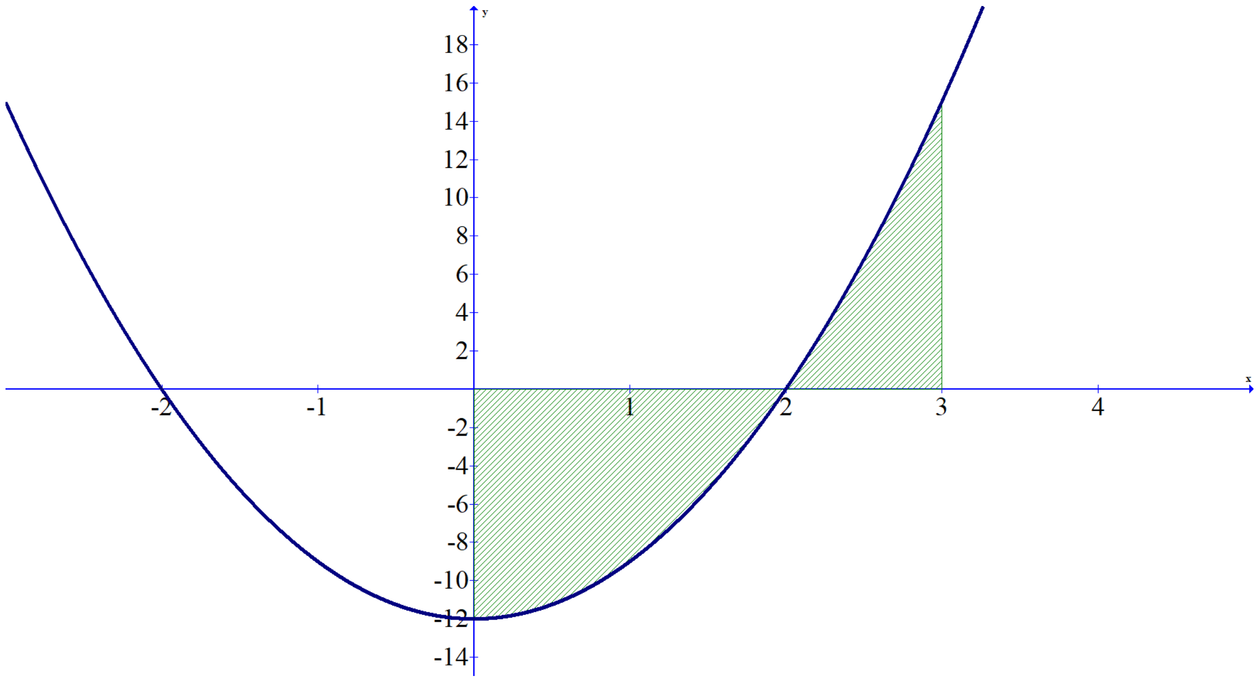
b) Use integration on your calculator to determine the area shaded between x = 0 and x = 2

Hint- this area will be negative. We need to make it positive using absolute values.



Answer Part b: $\left|\int_0^2 (3x^2 - 12)dx\right| = |-16| = 16$

c) Find the total shaded area



Add the areas computed in parts a and b to get the total shaded area.

Shaded area = $\int_2^3 (3x^2 - 12)dx + \left|\int_0^2 (3x^2 - 12)dx\right| =$

$$7 + |-16| = 7 + 16 = 23$$

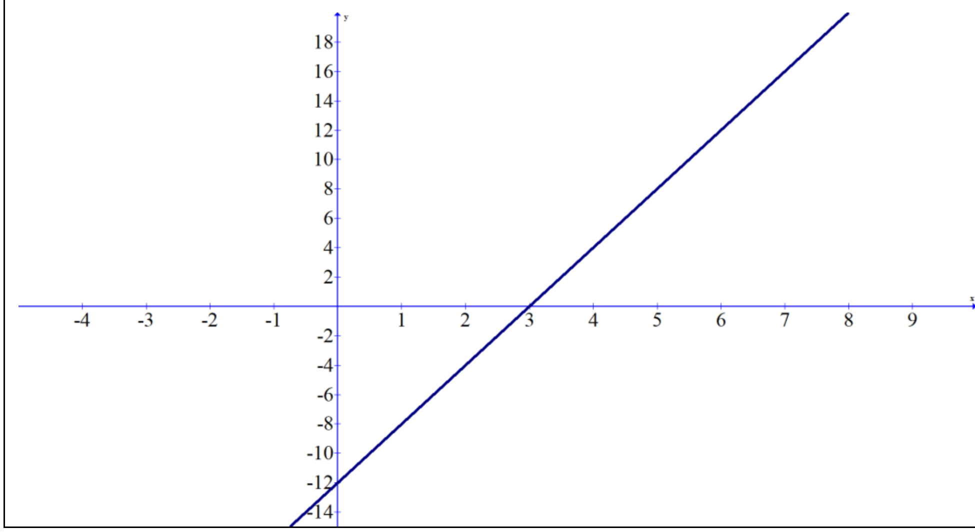
Answer part c: total shaded area 23 square units

Example: Let $f(x) = 4x - 12; [2,5]$

a) Sketch a graph of the function $f(x)$ over the given interval $[a, b]$.
b) Find any x-intercept within the interval $[a, b]$.
c) Find the total area between the x-axis and $f(x)$ over the interval $[a, b]$ using definite integrals.

a) Sketch a graph of the function $f(x)$ over the given interval $[a, b]$.

You may borrow a graph from your calculator



b) Find any x-intercept within the interval $[a, b]$.

$4x - 12 = 0$
$4x = 12$
$x = 3$
$x - intercept\ (3,0)$

This helps us determine the portion of the graph that is below the x-axis and above the x-axis.

c) Find the area between the x-axis and f(x) over the interval $[a, b]$ using definite integrals.

First, I will find the positive area by computing this integral by hand. The graph is above the x-axis in between $x = 3\ and\ x = 5\ in\ the\ given\ interval\ (2,5)$

$$\int_3^5 (4x - 12)dx = \int_3^5 (4x)dx - \int_3^5 12dx = 4\int_3^5 xdx - \int_3^5 12dx$$

$$= 4 * \frac{1}{2}x^2 - 12x\ \Big|_3^5$$

$$= 2x^2 - 12x\ \Big|_3^5$$

Evaluate at 5 then at 3 and subtract the results

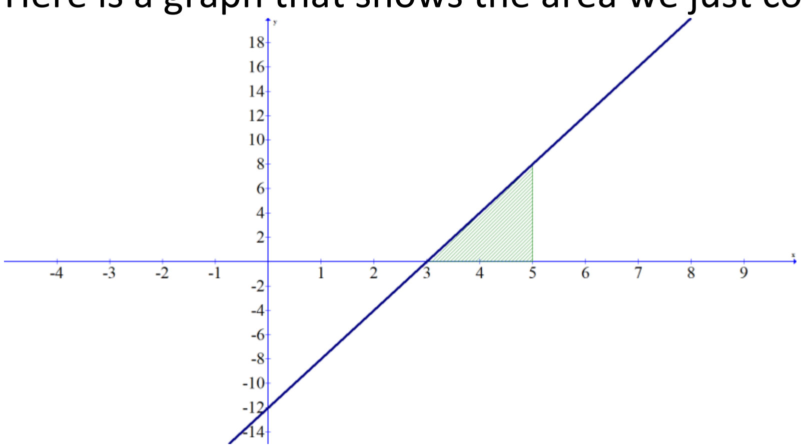
$2(5)^2 - 12(5) = 2 * 25 - 12(5) = 50 - 60 = -10$

$2(3)^2 - 12(3) = 2 * 9 - 12(3) = 18 - 36 = -18$

$$\int_3^5 (4x - 12)dx = -10 - (-18) = -10 + 18 = 8$$

The area above the x-axis: $\int_3^5 (4x - 12)dx = 8$

Here is a graph that shows the area we just computed:



c) continued

Next, I will find the “negative” area by computing this integral by hand.

I will use absolute values in my computation to remind me to make the answer positive.

The graph is beneath the x-axis between $x = 2 \; and \; x = 3 \; in \; the \; given \; interval \; (2,5)$

$$\left|\int_2^3 (4x - 12)dx\right| = \left|\int_2^3 4xdx - \int_2^3 12dx\right| = \left|4\int_2^3 4xdx - \int_2^3 12dx\right|$$

$$= \left|4 * \frac{1}{2}x^2 - 12x \;\; \Big|_2^3\right|$$

$$= \left|2x^2 - 12x \;\; \Big|_2^3\right|$$

Evaluate at 3 then at 2 and subtract the results

$2(3)^2 - 12(3) = 2 * 9 - 12(3) = 18 - 36 = \; -18$

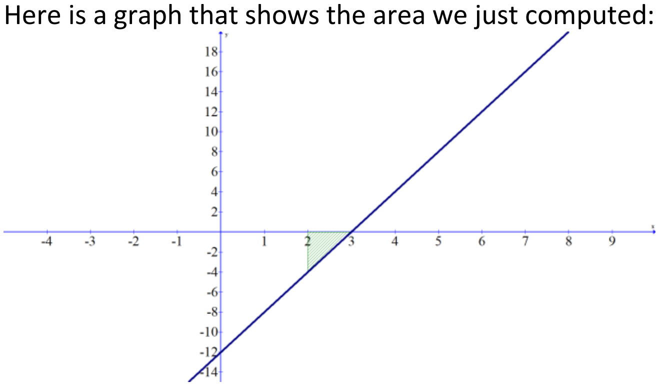
$2(2)^2 - 12(2) = 2 * 4 - 12(2) = 8 - 24 = \; -16$

$$\left|\int_2^3 (4x - 12)dx\right| = |-18 - (-16)| = |-18 + 16| = |-2|$$

$= -10 - (-18) = -10 + 18 = 8$ =2

The area beneath the x-axis: $\left|\int_2^3 (4x - 12)dx\right| = 2$

Here is a graph that shows the area we just computed:



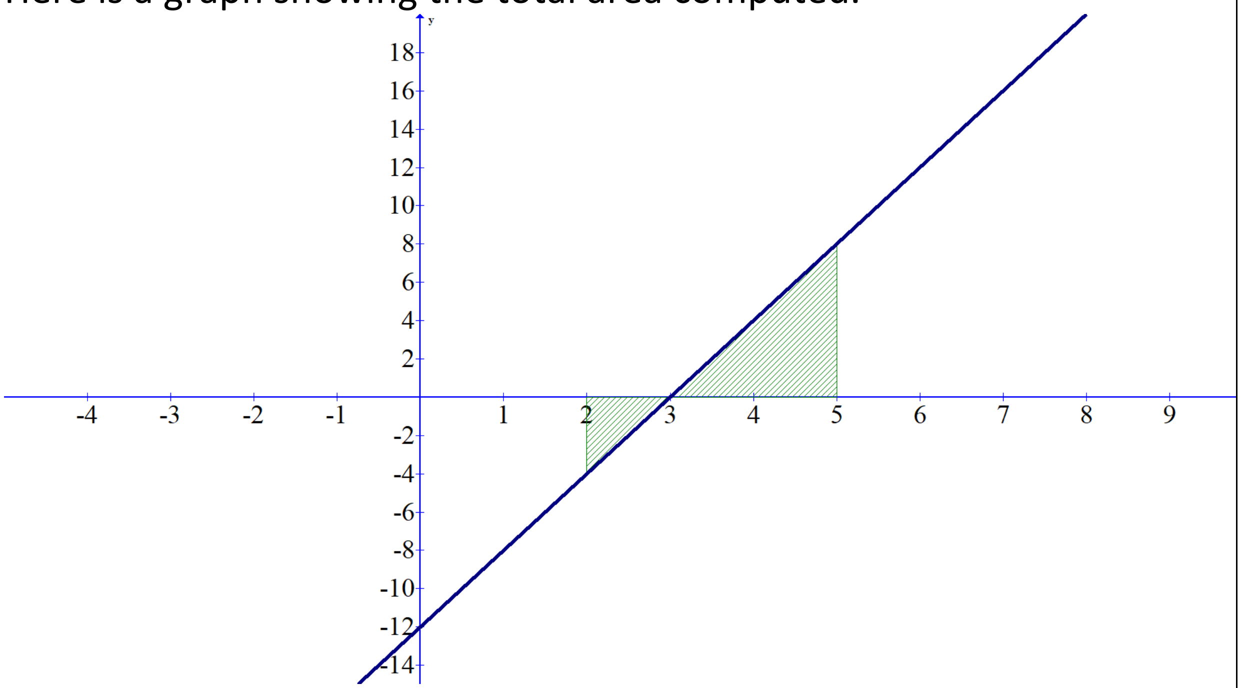
c) concluded:

Find the area between the x-axis and f(x) over the interval $[a, b]$ using definite integrals.

Total area = $\int_3^5 (4x - 12)dx + \left|\int_2^3 (4x - 12)dx\right| = 8 + 2 = 10$

Answer part c: shaded area is 10 square units.

Here is a graph showing the total area computed.

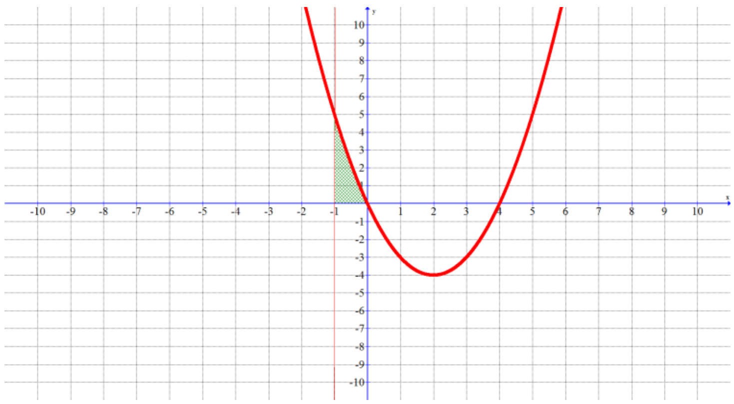


Homework:

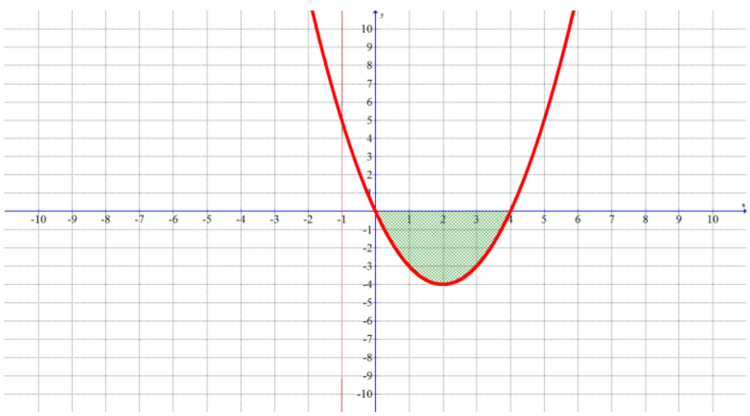
You may use your calculator to find the areas that are needed in problems 1- 4. (round to 2 decimals when needed)

1) The function $f(x) = x^2 - 4x$ *is graphed below.*

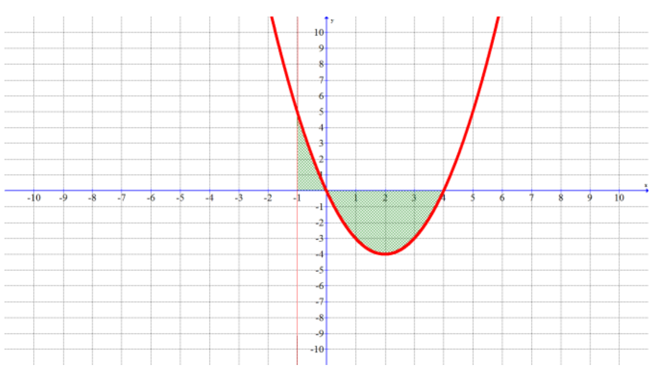
1a) Use integration on your calculator to determine the area shaded below between x = -1 and x = 0



1b) Use integration on your calculator to determine the area shaded below between x = 0 and x = 4

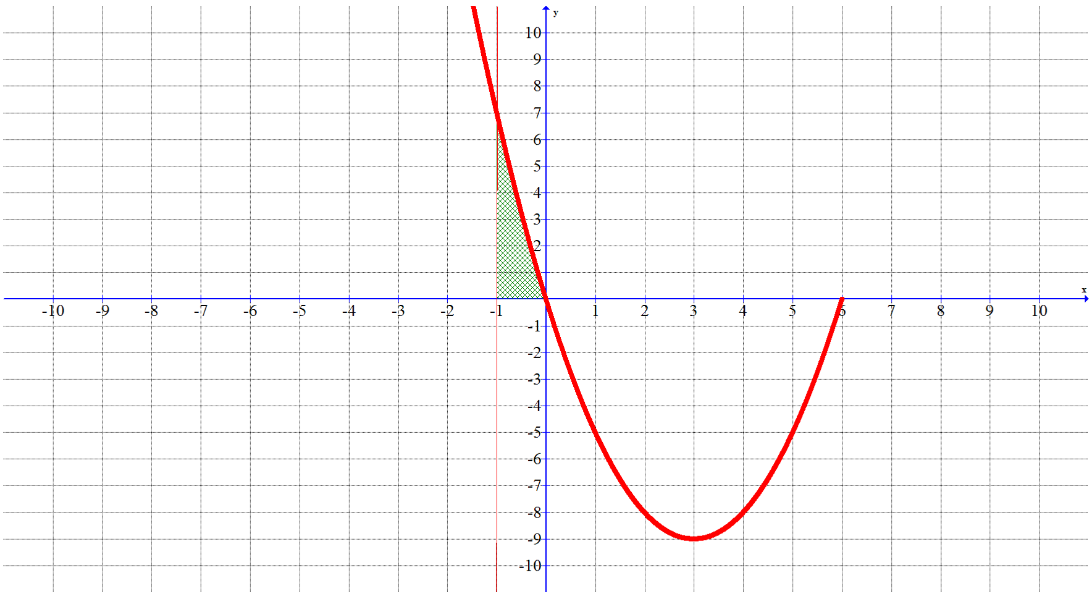


1c) Use integration on your calculator to determine the area shaded below between x = -1 and x = 4

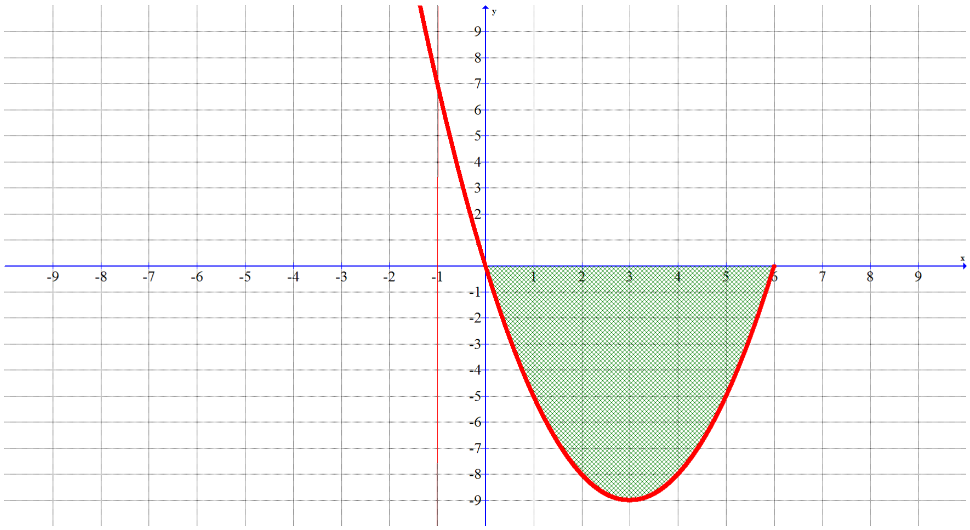


2) The function $f(x) = x^2 - 6x$ is graphed below.

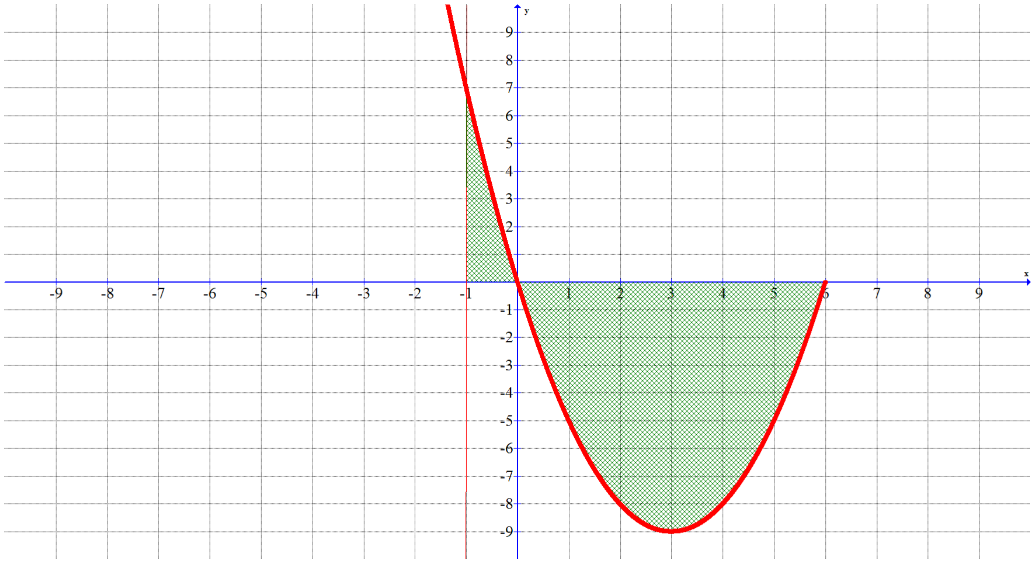
2a) Use integration on your calculator to determine the area shaded below between x = -1 and x = 0



2b) Use integration on your calculator to determine the area shaded below between x = 0 and x = 6



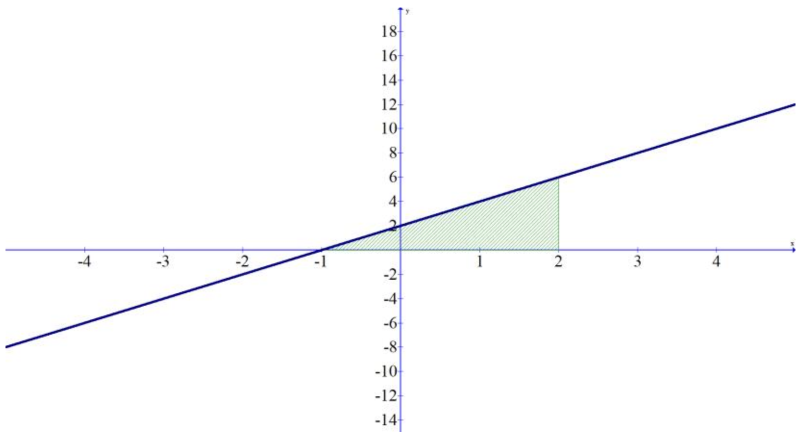
2c) Use integration on your calculator to determine the area shaded below between x = -1 and x = 6



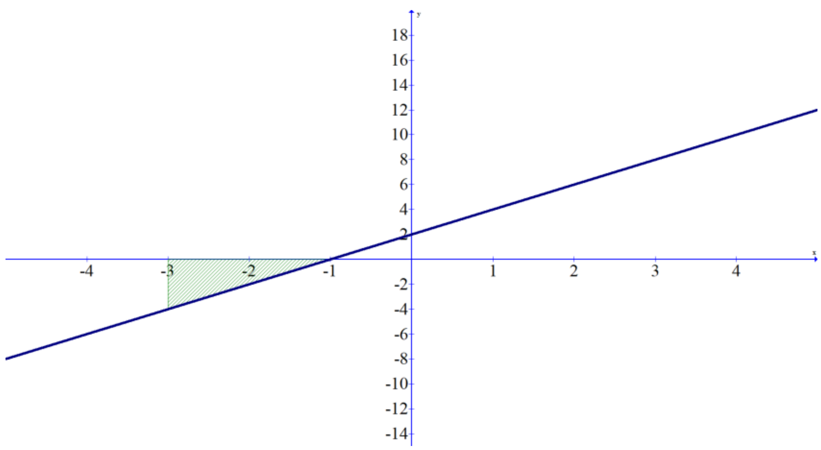
$answers$: $2a)$ 3.33 $\quad 2b)$ 36 $\quad 2c)$ 39.33

3) The function $f(x) = 2x + 2$ is graphed below

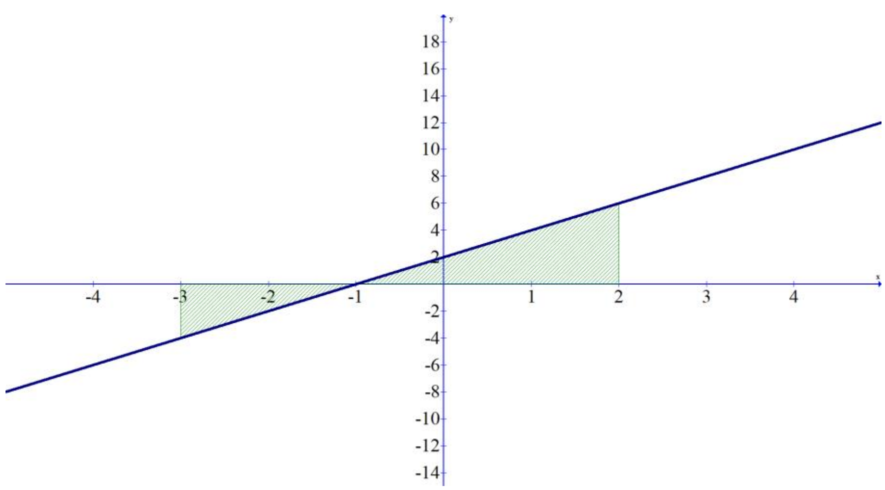
3a) Use integration on your calculator to determine the area shaded below between x = -1 and x = 2



3b) Use integration on your calculator to determine the area shaded below between x = -3 and x = -1

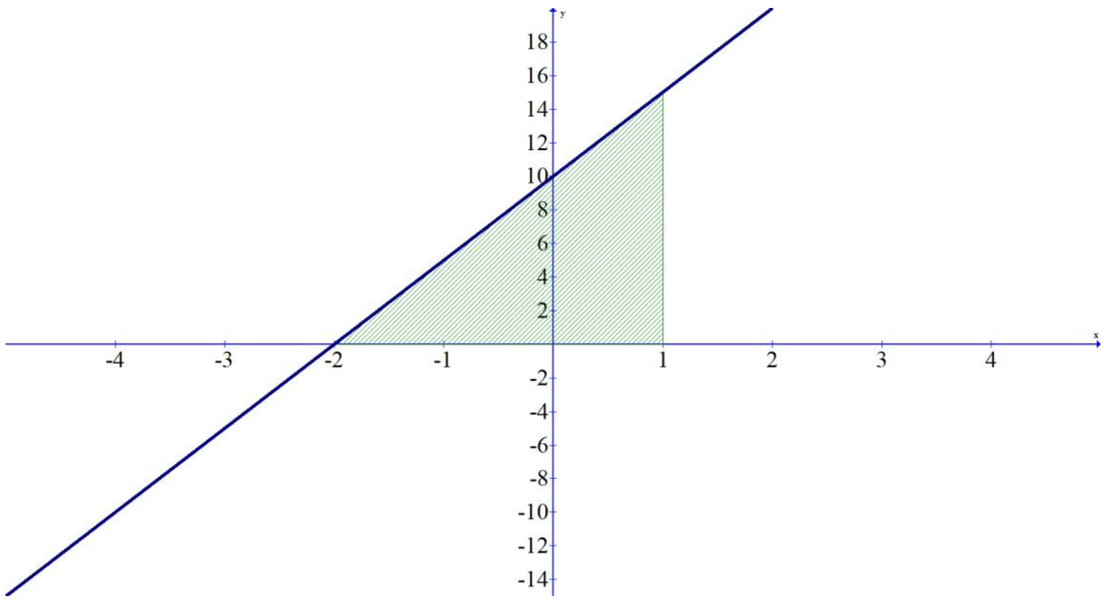


3c) Use integration on your calculator to determine the area shaded below between x = -3 and x = 2

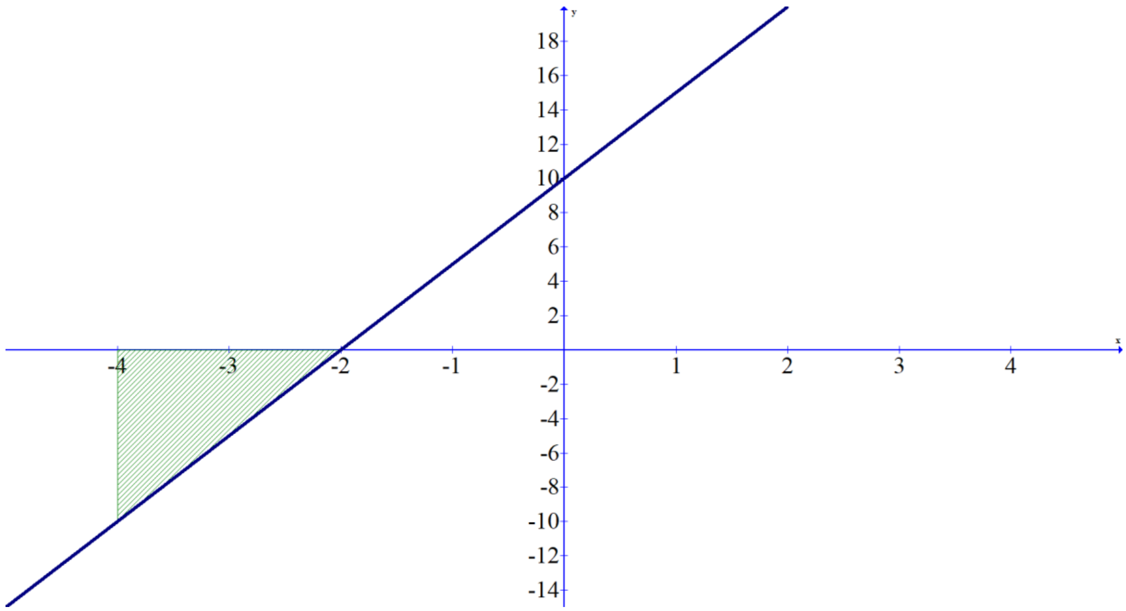


4) The function $f(x) = 5x + 10$ is graphed below

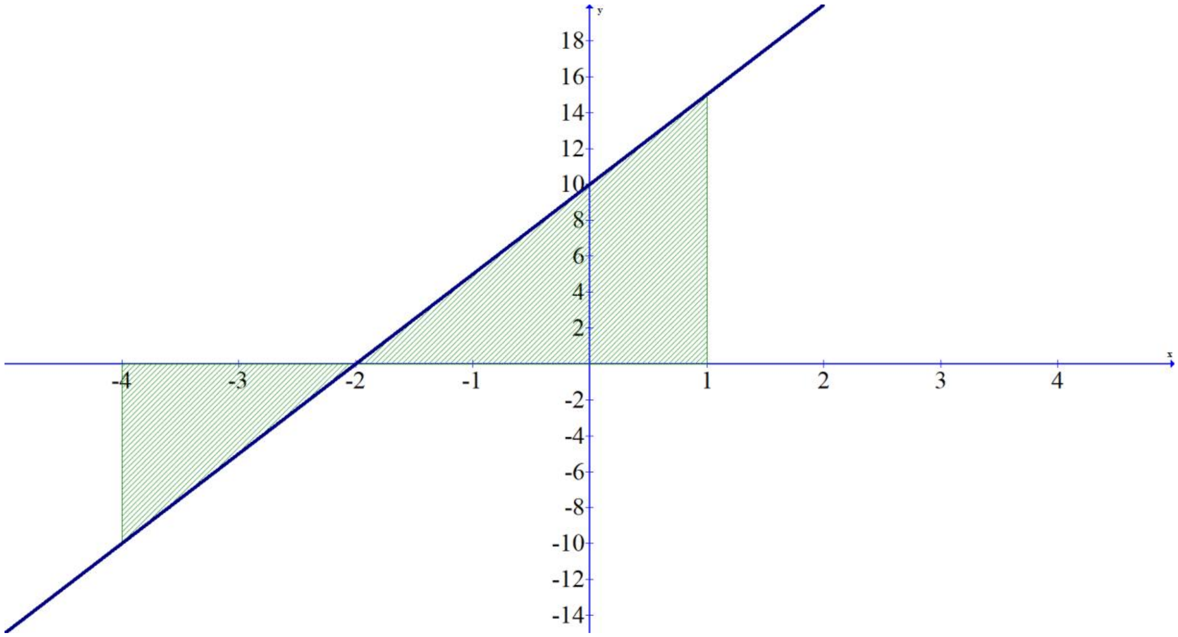
4a) Use integration on your calculator to determine the area shaded below between x = -2 and x = 1



4b) Use integration on your calculator to determine the area shaded below between x = -4 and x = -2



4c) Use integration on your calculator to determine the area shaded below between x = -4 and x = 1



Answers: $2a)$ 22.5 $2b)$ 10 $2c)$ 32.5

#5 – 12:

a) Sketch a graph of the function $f(x)$ over the given interval $[a, b]$.

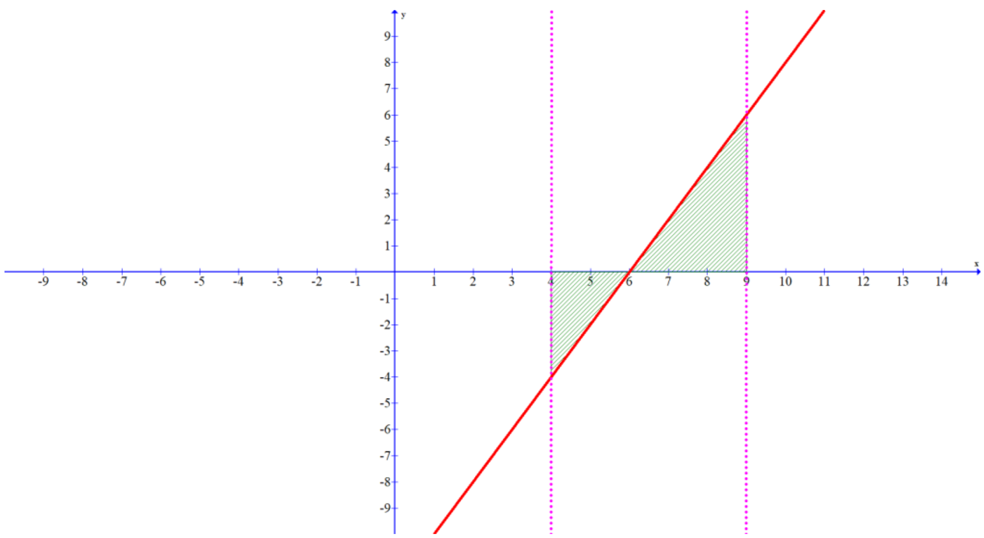
b) Find any x-intercept within the interval $[a, b]$.

c) Find the area between the x-axis and f(x) over the interval $[a, b]$ using definite integrals.

5) $f(x) = 2x - 14$; $[4,8]$

6) $f(x) = 2x - 12$; $[4,9]$

6a) Sketch a graph of the function $f(x)$ over the given interval $[a, b]$.



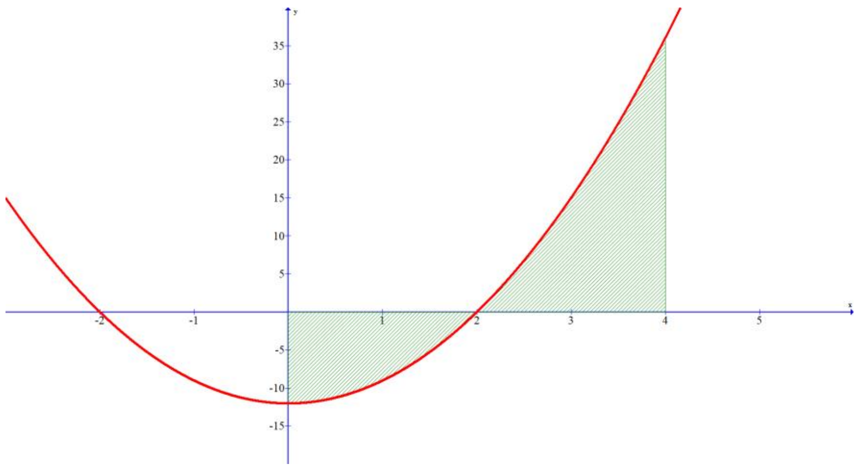
6b)

6c) Answer: $4 + 9 = 15$

7) $f(x) = 3x^2 - 3;\ [0,3]$

8) $f(x) = 3x^2 - 12;\ [0,4]$

8a) Sketch a graph of the function $f(x)$ over the given interval $[a, b]$.



8b) Find any x-intercept within the interval $[a, b]$.

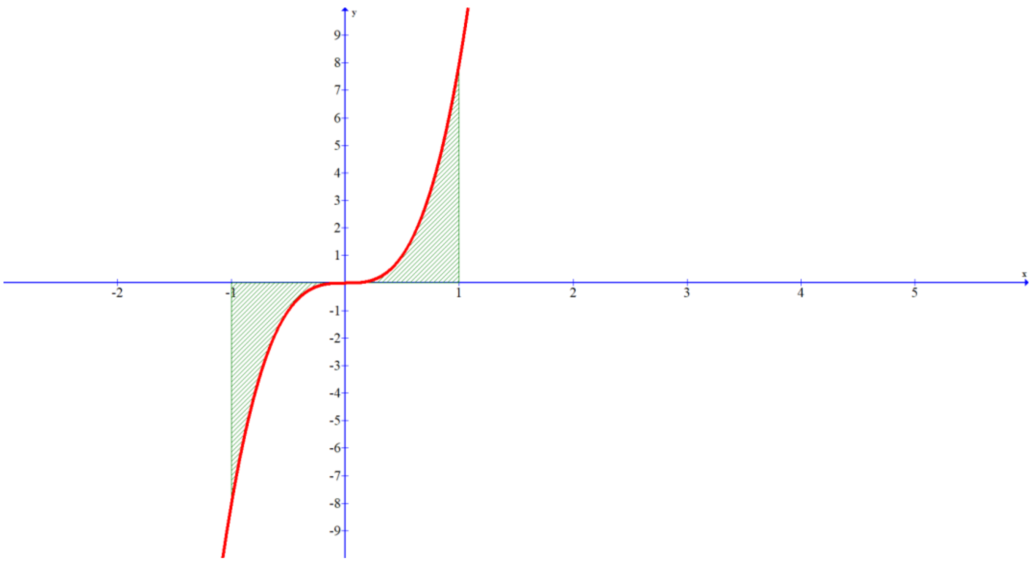
8c) Find the area between the x-axis and f(x) over the interval $[a, b]$ using definite integrals.

Answer: $|-16| + 32 = 16 + 32 = 48$

9) $f(x) = 4x^3$; $[-2,1]$

10) $f(x) = 8x^3$; $[-1,1]$

10a) Sketch a graph of the function $f(x)$ over the given interval $[a, b]$.



10b) Find any x-intercept within the interval $[a, b]$.

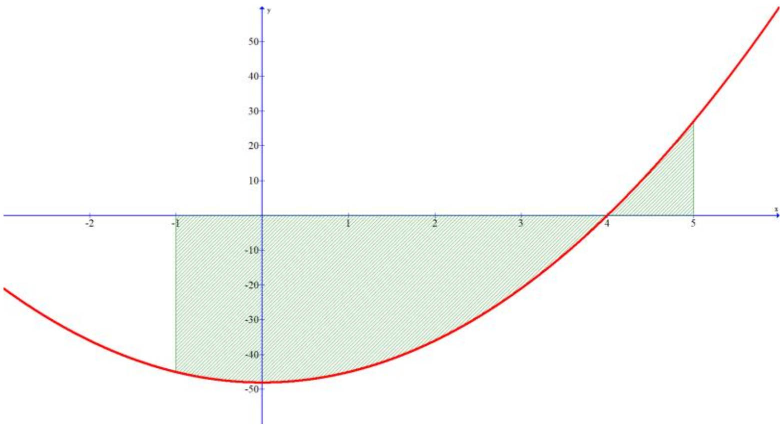
10c) Find the area between the x-axis and f(x) over the interval $[a, b]$ using definite integrals

Answer: $|-2| + 2 = 2 + 2 = 4$

11) $f(x) = 3x^2 - 27; [-1,5]$

12) $f(x) = 3x^2 - 48; \quad [-1,5]$

12a) Sketch a graph of the function $f(x)$ over the given interval $[a, b]$.



12b) Find any x-intercept within the interval $[a, b]$.

12c) Find the area between the x-axis and f(x) over the interval $[a, b]$ using definite integrals

Answer: $|-175| + 13 = 175 + 13 = 188$